Solution- Theoretical Question 1

A Swing with a Falling Weight

Part A

(a) Since the length of the string $L = s + R\theta$ is constant, its rate of change must be zero. Hence we have

$$\dot{s} + R\dot{\theta} = 0 \qquad \text{(A1)*}^{1}$$

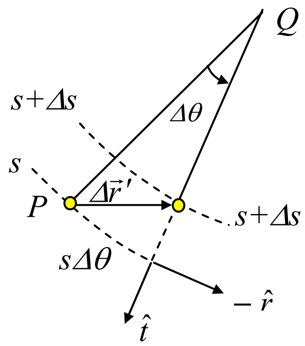
(b) Relative to O, Q moves on a circle of radius R with angular velocity $\dot{\theta}$, so

$$\vec{v}_Q = R\dot{\theta}\,\hat{t} = -\dot{s}\,\hat{t} \qquad \text{(A2)*}$$

(c) Refer to Fig. A1. Relative to Q, the displacement of P in a time interval Δt is $\Delta\vec{r}\,' = (s\Delta\theta)(-\hat{r}) + (\Delta s)\hat{t} = [(s\dot{\theta})(-\hat{r}) + \dot{s}\,\hat{t}]\Delta t$. It follows

$$\vec{v}\,' = -s\dot{\theta}\,\hat{r} + \dot{s}\,\hat{t} \qquad \text{(A3)*}$$

Figure A1



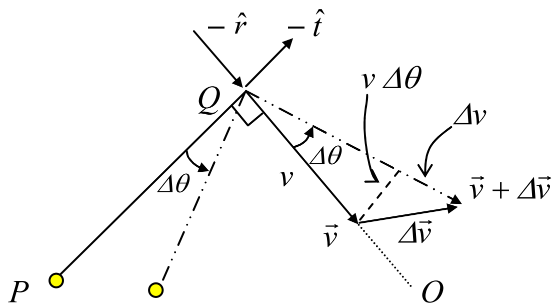
(d) The velocity of the particle relative to O is the sum of the two relative velocities given in Eqs. (A2) and (A3) so that

$$\vec{v} = \vec{v}\,' + \vec{v}_Q = (-s\dot{\theta}\,\hat{r} + \dot{s}\,\hat{t}\,) + R\dot{\theta}\,\hat{t} = -s\dot{\theta}\,\hat{r} \qquad \text{(A4)*}$$

(e) Refer to Fig. A2. The $(-\hat{t}\,)$-component of the velocity change $\Delta\vec{v}$ is given by $(-\hat{t}) \cdot \Delta\vec{v} = v\Delta\theta = v\dot{\theta}\,\Delta t$. Therefore, the $\hat{t}$-component of the acceleration $\vec{a} = \Delta\vec{v}\,/\,\Delta t$ is given by $\hat{t} \cdot \hat{a} = -v\dot{\theta}$. Since the speed v of the particle is $s\dot{\theta}$ according to Eq. (A4), we see that the $\hat{t}$-component of the particle's acceleration at P is given by

$$\vec{a} \cdot \hat{t} = -v\dot{\theta} = -(s\dot{\theta})\dot{\theta} = -s\dot{\theta}^2 \qquad \text{(A5)*}$$

Figure A2



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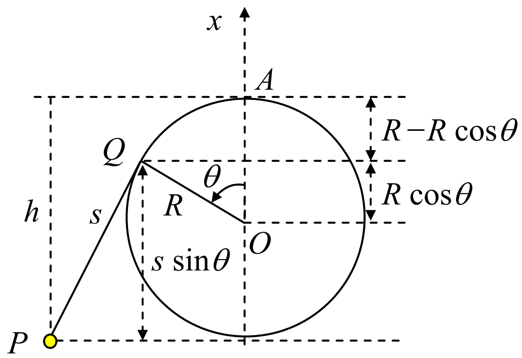
[1] An equation marked with an asterisk contains answer to the problem.

Note that, from Fig. A2, the radial component of the acceleration may also be obtained as $\vec{a}\cdot\hat{r} = -dv/dt = -d(s\dot{\theta})/dt$.

(f) Refer to Fig. A3. The gravitational potential energy of the particle is given by $U = -mgh$. It may be expressed in terms of s and θ as

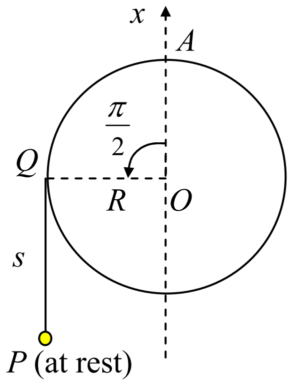
$$U(\theta) = -mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(A6)*}$$

Figure A3



(g) At the lowest point of its trajectory, the particle's gravitational potential energy U must assume its minimum value U_m. If the particle's mechanical energy E were equal to U_m, its kinetic energy would be zero. The particle would then remain stationary and be in the static equilibrium state shown in Fig. A4. Thus, the potential energy reaches its minimum value when $\theta = \pi/2$ or $s = L - \pi R/2$.

Figure A4



From Fig. A4 or Eq. (A6), the minimum potential energy is then

$$U_m = U(\frac{\pi}{2}) = -mg[R + L - (\pi R/2)]. \qquad \text{(A7)}$$

Initially, the total mechanical energy E is 0. Since E is conserved, the speed v_m of the particle at the lowest point of its trajectory must satisfy

$$E = 0 = \frac{1}{2}mv_m^2 + U_m. \qquad \text{(A8)}$$

From Eqs. (A7) and (A8), we obtain

$$v_m = \sqrt{-2U_m/m} = \sqrt{2g[R + (L - \pi R/2)]}. \qquad \text{(A9)*}$$

Part B

(h) From Eq. (A6), the total mechanical energy of the particle may be written as

$$E = 0 = \frac{1}{2}mv^2 + U(\theta) = \frac{1}{2}mv^2 - mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(B1)}$$

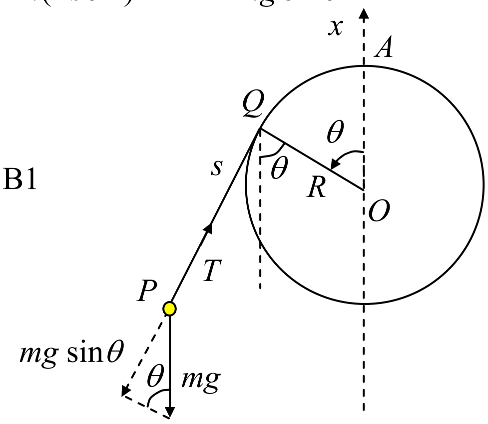
From Eq. (A4), the speed v is equal to $s\dot{\theta}$. Therefore, Eq. (B1) implies

$$v^2 = (s\dot{\theta})^2 = 2g[R(1-\cos\theta) + s\sin\theta] \qquad \text{(B2)}$$

Let T be the tension in the string. Then, as Fig. B1 shows, the $\hat{t}$-component of the net force on the particle is $-T + mg\sin\theta$. From Eq. (A5), the tangential acceleration of the particle is $(-s\dot{\theta}^2)$. Thus, by Newton's second law, we have

$$m(-s\dot{\theta}^2) = -T + mg\sin\theta \qquad \text{(B3)}$$

Figure B1



According to the last two equations, the tension may be expressed as

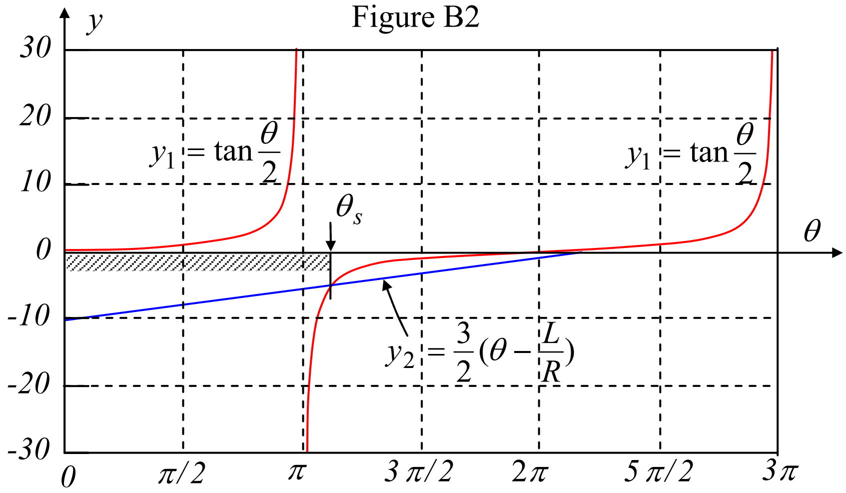
$$T = m(s\dot{\theta}^2 + g\sin\theta) = \frac{mg}{s}[2R(1-\cos\theta) + 3s\sin\theta]$$

$$= \frac{2mgR}{s}[\tan\frac{\theta}{2} - \frac{3}{2}(\theta - \frac{L}{R})](\sin\theta) \qquad \text{(B4)}$$

$$= \frac{2mgR}{s}(y_1 - y_2)(\sin\theta)$$

The functions $y_1 = \tan(\theta/2)$ and $y_2 = 3(\theta - L/R)/2$ are plotted in Fig B2.

Figure B2



From Eq. (B4) and Fig. B2, we obtain the result shown in Table B1. The angle at which .$y_2 = y_1$ is called θ_s ($\pi < \theta_s < 2\pi$) and is given by

$$\frac{3}{2}(\theta_s - \frac{L}{R}) = \tan\frac{\theta_s}{2} \tag{B5}$$

or, equivalently, by

$$\frac{L}{R} = \theta_s - \frac{2}{3}\tan\frac{\theta_s}{2} \tag{B6}$$

Since the ratio L/R is known to be given by

$$\frac{L}{R} = \frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} = (\pi + \frac{\pi}{8}) - \frac{2}{3}\tan\frac{1}{2}(\pi + \frac{\pi}{8}) \tag{B7}$$

one can readily see from the last two equations that $\theta_s = 9\pi/8$.

Table B1

	$(y_1 - y_2)$	$\sin\theta$	tension T
$0 < \theta < \pi$	positive	positive	positive
$\theta = \pi$	$+\infty$	0	positive
$\pi < \theta < \theta_s$	negative	negative	positive
$\theta = \theta_s$	zero	negative	zero
$\theta_s < \theta < 2\pi$	positive	negative	negative

Table B1 shows that the tension T must be positive (or the string must be taut and straight) in the angular range $0 < \theta < \theta_s$. Once θ reaches θ_s, the tension T becomes zero and the part of the string not in contact with the rod will not be straight afterwards. The shortest possible value $s_{\min}$ for the length s of the line segment QP therefore occurs at $\theta = \theta_s$ and is given by

$$s_{\min} = L - R\theta_s = R(\frac{9\pi}{8} + \frac{2}{3}\cot\frac{\pi}{16} - \frac{9\pi}{8}) = \frac{2R}{3}\cot\frac{\pi}{16} = 3.352R \tag{B8}$$

When $\theta = \theta_s$, we have $T = 0$ and Eqs. (B2) and (B3) then leads to $v^2 = -gs\sin\theta$. Hence the speed v_s is

$$v_s = \sqrt{-g s_{\min} \sin\theta_s} = \sqrt{\frac{2gR}{3}\cot\frac{\pi}{16}\sin\frac{\pi}{8}} = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16} \qquad \text{(B9)*}$$

$$= 1.133\sqrt{gR}$$

(i) When $\theta \geq \theta_s$, the particle moves like a projectile under gravity. As shown in Fig. B3, it is projected with an initial speed v_s from the position $P = (x_s, y_s)$ in a direction making an angle $\phi = (3\pi/2 - \theta_s)$ with the y-axis.

The speed v_H of the particle at the highest point of its parabolic trajectory is equal to the y-component of its initial velocity when projected. Thus,

$$v_H = v_s \sin(\theta_s - \pi) = \sqrt{\frac{4gR}{3}}\cos\frac{\pi}{16}\sin\frac{\pi}{8} = 0.4334\sqrt{gR} \qquad \text{(B10)*}$$

The horizontal distance H traveled by the particle from point P to the point of maximum height is

$$H = \frac{v_s^2 \sin 2(\theta_s - \pi)}{2g} = \frac{v_s^2}{2g}\sin\frac{9\pi}{4} = 0.4535R \qquad \text{(B11)}$$

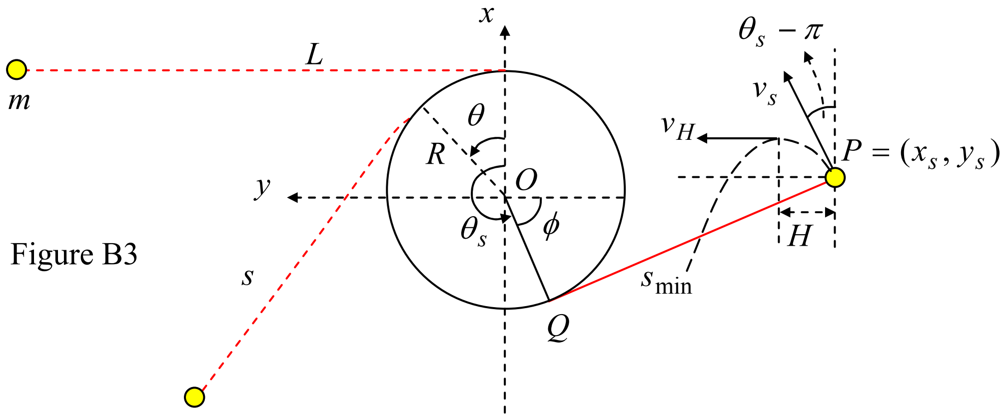


Figure B3

The coordinates of the particle when $\theta = \theta_s$ are given by

$$x_s = R\cos\theta_s - s_{\min}\sin\theta_s = -R\cos\frac{\pi}{8} + s_{\min}\sin\frac{\pi}{8} = 0.358R \qquad \text{(B12)}$$

$$y_s = R\sin\theta_s + s_{\min}\cos\theta_s = -R\sin\frac{\pi}{8} - s_{\min}\cos\frac{\pi}{8} = -3.478R \qquad \text{(B13)}$$

Evidently, we have $|y_s| > (R + H)$. Therefore the particle can indeed reach its maximum height without striking the surface of the rod.

Part C

(j) Assume the weight is initially lower than O by h as shown in Fig. C1.

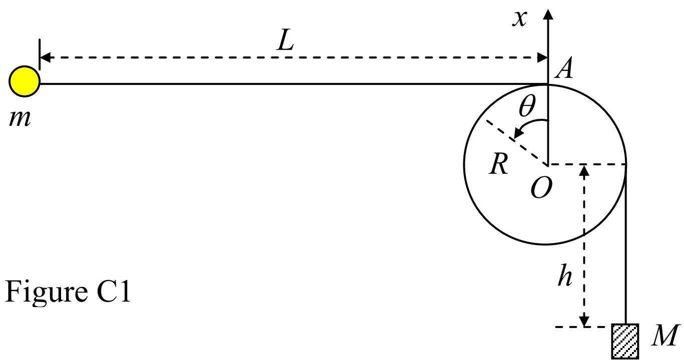


Figure C1

When the weight has fallen a distance D and stopped, the law of conservation of total mechanical energy as applied to the particle-weight pair as a system leads to

$$-Mgh = E' - Mg(h+D) \qquad \text{(C1)}$$

where E' is the *total mechanical energy of the particle* when the weight has stopped. It follows

$$E' = MgD \qquad \text{(C2)}$$

Let Λ be the total length of the string. Then, its value at $\theta = 0$ must be the same as at any other angular displacement θ. Thus we must have

$$\Lambda = L + \frac{\pi}{2}R + h = s + R(\theta + \frac{\pi}{2}) + (h+D) \qquad \text{(C3)}$$

Noting that $D = \alpha L$ and introducing $\ell = L - D$, we may write

$$\ell = L - D = (1-\alpha)L \qquad \text{(C4)}$$

From the last two equations, we obtain

$$s = L - D - R\theta = \ell - R\theta \qquad \text{(C5)}$$

After the weight has stopped, the total mechanical energy of the particle must be conserved. According to Eq. (C2), we now have, instead of Eq. (B1), the following equation:

$$E' = MgD = \frac{1}{2}mv^2 - mg[R(1-\cos\theta) + s\sin\theta] \qquad \text{(C6)}$$

The square of the particle's speed is accordingly given by

$$v^2 = (s\dot{\theta})^2 = \frac{2MgD}{m} + 2gR[(1-\cos\theta) + \frac{s}{R}\sin\theta] \quad . \qquad \text{(C7)}$$

Since Eq. (B3) stills applies, the tension T of the string is given by

$$-T + mg\sin\theta = m(-s\dot{\theta}^2) \qquad \text{(C8)}$$

From the last two equations, it follows

$$\begin{aligned} T &= m(s\dot{\theta}^2 + g\sin\theta) \\ &= \frac{mg}{s}[\frac{2M}{m}D + 2R(1-\cos\theta) + 3s\sin\theta] \qquad \text{(C9)} \\ &= \frac{2mgR}{s}[\frac{MD}{mR} + (1-\cos\theta) + \frac{3}{2}(\frac{\ell}{R} - \theta)\sin\theta] \end{aligned}$$

where Eq. (C5) has been used to obtain the last equality.

We now introduce the function

$$f(\theta) = 1 - \cos\theta + \frac{3}{2}(\frac{\ell}{R} - \theta)\sin\theta \qquad \text{(C10)}$$

From the fact $\ell = (L-D) >> R$, we may write

$$f(\theta) \approx 1 + \frac{3}{2}\frac{\ell}{R}\sin\theta - \cos\theta = 1 + A\sin(\theta - \phi) \tag{C11}$$

where we have introduced

$$A = \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^2}\ , \qquad \phi = \tan^{-1}\frac{\frac{3\ell}{2R}}{\sqrt{1 + (\frac{3\ell}{2R})^2}} \tag{C12}$$

From Eq. (C11), the minimum value of $f(\theta)$ is seen to be given by

$$f_{\min} = 1 - A = 1 - \sqrt{1 + (\frac{3}{2}\frac{\ell}{R})^2} \tag{C13}$$

Since the tension T remains nonnegative as the particle swings around the rod, we have from Eq. (C9) the inequality

$$\frac{MD}{mR} + f_{\min} = \frac{M(L-\ell)}{mR} + 1 - \sqrt{1 + (\frac{3\ell}{2R})^2} \geq 0 \tag{C14}$$

or

$$(\frac{ML}{mR}) + 1 \ \geq \ (\frac{M\ell}{mR}) + \sqrt{1 + (\frac{3\ell}{2R})^2} \approx (\frac{M\ell}{mR}) + (\frac{3\ell}{2R}) \tag{C15}$$

From Eq. (C4), Eq. (C15) may be written as

$$(\frac{ML}{mR}) + 1 \ \geq \ [(\frac{ML}{mR}) + (\frac{3L}{2R})](1 - \alpha) \tag{C16}$$

Neglecting terms of the order (R/L) or higher, the last inequality leads to

$$\alpha \ \geq \ 1 - \frac{(\frac{ML}{mR}) + 1}{(\frac{ML}{mR}) + (\frac{3L}{2R})} = \frac{(\frac{3L}{2R}) - 1}{(\frac{ML}{mR}) + (\frac{3L}{2R})} = \frac{1 - \frac{2R}{3L}}{\frac{2M}{3m} + 1} \approx \frac{1}{1 + \frac{2M}{3m}} \tag{C17}$$

The critical value for the ratio D/L is therefore

$$\alpha_c = \frac{1}{(1 + \frac{2M}{3m})} \tag{C18)*}$$

Solution- Theoretical Question 2

A Piezoelectric Crystal Resonator under an Alternating Voltage

Part A

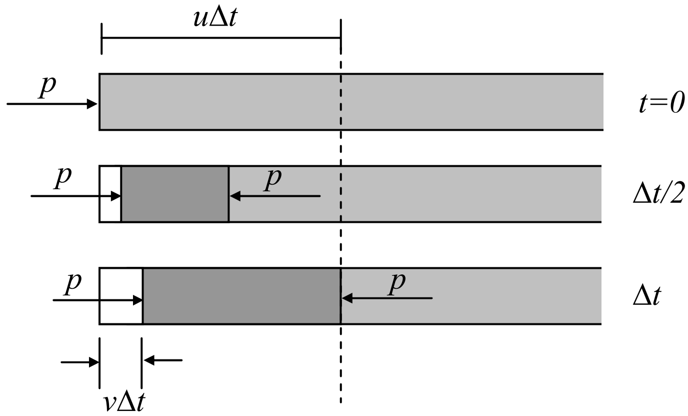
(a) Refer to Figure A1. The left face of the rod moves a distance $v\Delta t$ while the pressure wave travels a distance $u\Delta t$ with $u = \sqrt{Y/\rho}$. The strain at the left face is

$$S = \frac{\Delta\ell}{\ell} = \frac{-v\Delta t}{u\Delta t} = \frac{-v}{u} \tag{A1a)*[1]}$$

From Hooke's law, the pressure at the left face is

$$p = -YS = Y\frac{v}{u} = \rho uv \tag{A1b)*}$$

Figure A1



(b) The velocity v is related to the displacement ξ as in a simple harmonic motion (or a uniform circular motion, as shown in Figure A2) of angular frequency $\omega = ku$. Therefore, if $\xi(x,t) = \xi_0 \sin k(x - ut)$, then

$$v(x,t) = -ku\xi_0 \cos k(x - ut) . \tag{A2)*}$$

The strain and pressure are related to velocity as in Problem (a). Hence,

$$S(x,t) = -v(x,t)/u = k\xi_0 \cos k(x - ut) \tag{A3)*}$$

$$\begin{aligned} p(x,t) &= \rho uv(x,t) = -k\rho u^2 \xi_0 \cos k(x - ut) \\ &= -YS(x,t) = -kY\xi_0 \cos k(x - ut) \end{aligned} \tag{A4)*}$$

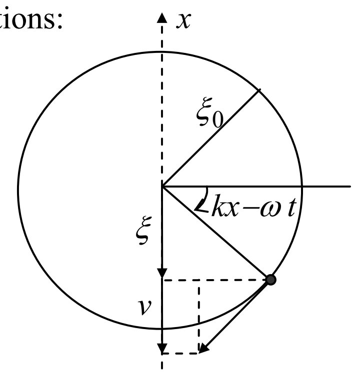
Alternatively, the answers may be obtained by differentiations:

$$v(x,t) = \frac{\Delta\xi}{\Delta t} = -ku\xi_0 \cos k(x - ut) ,$$

$$S(x,t) = \frac{\Delta\xi}{\Delta x} = k\xi_0 \cos k(x - ut) ,$$

$$p(x,t) = -Y\frac{\Delta\xi}{\Delta x} = -kY\xi_0 \cos k(x - ut) .$$

Figure A2



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[1] An equations marked with an asterisk contains answer to the problem.

Part B

(c) Since the angular frequency ω and speed of propagation u are given, the wavelength is given by $\lambda = 2\pi / k$ with $k = \omega / u$. The spatial variation of the displacement ξ is therefore described by

$$g(x) = B_1 \sin k(x - \frac{b}{2}) + B_2 \cos k(x - \frac{b}{2}) \qquad \text{(B1)}$$

Since the centers of the electrodes are assumed to be stationary, $g(b/2) = 0$. This leads to $B_2 = 0$. Given that the maximum of $g(x)$ is 1, we have $A = \pm 1$ and

$$g(x) = \pm \sin \frac{\omega}{u}(x - \frac{b}{2}) \qquad \text{(B2)*}$$

Thus, the displacement is

$$\xi(x,t) = \pm 2\xi_0 \sin \frac{\omega}{u}(x - \frac{b}{2}) \cos \omega t \qquad \text{(B3)}$$

(d) Since the pressure p (or stress T) must vanish at the end faces of the quartz slab (i.e., $x = 0$ and $x = b$), the answer to this problem can be obtained, by analogy, from the resonant frequencies of sound waves in an open pipe of length b. However, given that the centers of the electrodes are stationary, all even harmonics of the fundamental tone must be excluded because they have antinodes, rather than nodes, of displacement at the bisection plane of the slab.

Since the fundamental tone has a wavelength $\lambda = 2b$, the fundamental frequency is given by $f_1 = u/(2b)$. The speed of propagation u is given by

$$u = \sqrt{\frac{Y}{\rho}} = \sqrt{\frac{7.87 \times 10^{10}}{2.65 \times 10^{3}}} = 5.45 \times 10^{3} \text{ m/s} \qquad \text{(B4)}$$

and, given that $b = 1.00 \times 10^{-2}$ m, the two lowest standing wave frequencies are

$$f_1 = \frac{u}{2b} = 273 \text{ (kHz)}, \quad f_3 = 3f_1 = \frac{3u}{2b} = 818 \text{ (kHz)} \qquad \text{(B5)*}$$

[*Alternative solution to Problems (c) and (d)*]:

A longitudinal standing wave in the quartz slab has a displacement node at $x = b/2$. It may be regarded as consisting of two waves traveling in opposite directions. Thus, its displacement and velocity must have the following form

$$\begin{aligned} \xi(x,t) &= 2\xi_m \sin k(x - \frac{b}{2}) \cos \omega t \\ &= \xi_m [\sin k(x - \frac{b}{2} - ut) + \sin k(x - \frac{b}{2} + ut)] \end{aligned} \qquad \text{(B6)}$$

$$\begin{aligned} v(x,t) &= -ku\xi_m [\cos k(x - \frac{b}{2} - ut) - \cos k(x - \frac{b}{2} + ut)] \\ &= -2\omega\xi_m \sin k(x - \frac{b}{2}) \sin \omega t \end{aligned} \qquad \text{(B7)}$$

where $\omega = ku$ and the first and second factors in the square brackets represent waves traveling along the $+x$ and $-x$ directions, respectively. Note that Eq. (B6) is identical to Eq. (B3) if we set $\xi_m = \pm \xi_0$.

For a wave traveling along the –x direction, the velocity v must be replaced by –v in Eqs. (A1a) and (A1b) so that we have

$$S = \frac{-v}{u} \quad \text{and} \quad p = \rho u v \qquad \text{(waves traveling along } +x\text{)} \tag{B8}$$

$$S = \frac{v}{u} \quad \text{and} \quad p = -\rho u v \qquad \text{(waves traveling along } -x\text{)} \tag{B9}$$

As in Problem (b), the strain and pressure are therefore given by

$$\begin{aligned} S(x,t) &= -k\xi_m[-\cos k(x-\frac{b}{2}-ut) - \cos k(x-\frac{b}{2}+ut)] \\ &= 2k\xi_m \cos k(x-\frac{b}{2})\cos\omega t \end{aligned} \tag{B10}$$

$$\begin{aligned} p(x,t) &= -\rho u\omega\xi_m[\cos k(x-\frac{b}{2}-ut) + \cos k(x-\frac{b}{2}+ut)] \\ &= -2\rho u\omega\xi_m \cos k(x-\frac{b}{2})\cos\omega t \end{aligned} \tag{B11}$$

Note that v, S, and p may also be obtained by differentiating ξ as in Problem (b).

The stress T or pressure p must be zero at both ends ($x = 0$ and $x = b$) of the slab at all times because they are free. From Eq. (B11), this is possible only if $\cos(kb/2) = 0$ or

$$kb = \frac{\omega}{u}b = \frac{2\pi f}{\lambda f}b = n\pi, \qquad n = 1, 3, 5, \cdots \tag{B12}$$

In terms of wavelength λ, Eq. (B12) may be written as

$$\lambda = \frac{2b}{n}, \qquad n = 1, 3, 5, \cdots. \tag{B13}$$

The frequency is given by

$$f = \frac{u}{\lambda} = \frac{nu}{2b} = \frac{n}{2b}\sqrt{\frac{Y}{\rho}}, \qquad n = 1, 3, 5, \cdots. \tag{B14}$$

This is identical with the results given in Eqs. (B4) and (B5).

(e) From Eqs. (5a) and (5b) in the Question, the piezoelectric effect leads to the equations

$$T = Y(S - d_p E) \tag{B15}$$

$$\sigma = Yd_p S + \varepsilon_T(1 - Y\frac{d_p^2}{\varepsilon_T})E \tag{B16}$$

Because $x = b/2$ must be a node of displacement for any longitudinal standing wave in the slab, the displacement ξ and strain S must have the form given in Eqs. (B6) and (B10), i.e., with $\omega = ku$,

$$\xi(x,t) = \xi_m \sin k(x-\frac{b}{2})\cos(\omega t + \phi) \tag{B17}$$

$$S(x,t) = k\xi_m \cos k(x-\frac{b}{2})\cos(\omega t + \phi) \tag{B18}$$

where a phase constant ϕ is now included in the time-dependent factors.

By assumption, the electric field E between the electrodes is uniform and depends only on time:

$$E(x,t) = \frac{V(t)}{h} = \frac{V_m \cos\omega t}{h}. \qquad \text{(B19)}$$

Substituting Eqs. (B18) and (B19) into Eq. (B15), we have

$$T = Y[k\xi_m \cos k(x-\frac{b}{2})\cos(\omega t+\phi) - \frac{d_p}{h}V_m \cos\omega t] \qquad \text{(B20)}$$

The stress T must be zero at both ends ($x = 0$ and $x = b$) of the slab at all times because they are free. This is possible only if $\phi = 0$ and

$$k\xi_m \cos\frac{kb}{2} = d_p \frac{V_m}{h} \qquad \text{(B21)}$$

Since $\phi = 0$, Eqs. (B16), (B18), and (B19) imply that the surface charge density must have the same dependence on time t and may be expressed as

$$\sigma(x,t) = \sigma(x)\cos\omega t \qquad \text{(B22)}$$

with the dependence on x given by

$$\begin{aligned}\sigma(x) &= Yd_p k\xi_m \cos k(x-\frac{b}{2}) + \varepsilon_T(1-Y\frac{d_p^2}{\varepsilon_T})\frac{V_m}{h} \\ &= [Y\frac{d_p^2}{\cos\frac{kb}{2}}\cos k(x-\frac{b}{2}) + \varepsilon_T(1-Y\frac{d_p^2}{\varepsilon_T})]\frac{V_m}{h}\end{aligned} \qquad \text{(B23)*}$$

(f) At time t, the total surface charge $Q(t)$ on the lower electrode is obtained by integrating $\sigma(x,t)$ in Eq. (B22) over the surface of the electrode. The result is

$$\begin{aligned}\frac{Q(t)}{V(t)} &= \frac{1}{V(t)}\int_0^b \sigma(x,t)\,w\,dx = \frac{1}{V_m}\int_0^b \sigma(x)\,w\,dx \\ &= \frac{w}{h}\int_0^b [Y\frac{d_p^2}{\cos\frac{kb}{2}}\cos k(x-\frac{b}{2}) + \varepsilon_T(1-Y\frac{d_p^2}{\varepsilon_T})]dx \\ &= (\varepsilon_T\frac{bw}{h})[Y\frac{d_p^2}{\varepsilon_T}(\frac{2}{kb}\tan\frac{kb}{2}) + (1-Y\frac{d_p^2}{\varepsilon_T})] \\ &= C_0[\alpha^2(\frac{2}{kb}\tan\frac{kb}{2}) + (1-\alpha^2)]\end{aligned} \qquad \text{(B24)}$$

where

$$C_0 = \varepsilon_T\frac{bw}{h}, \qquad \alpha^2 = Y\frac{d_p^2}{\varepsilon_T} = \frac{(2.25)^2\times 10^{-2}}{1.27\times 4.06} = 9.82\times 10^{-3} \qquad \text{(B25)*}$$

(The constant α is called the *electromechanical coupling coefficient*.)

Note: The result $C_0 = \varepsilon_T bw/h$ can readily be seen by considering the static limit $k = 0$ of Eq. (5) in the Question. Since $\tan x \approx x$ when $x << 1$, we have

$$\lim_{k\to 0} Q(t)/V(t) \approx C_0[\alpha^2 + (1-\alpha^2)] = C_0 \qquad \text{(B26)}$$

Evidently, the constant C_0 is the capacitance of the parallel-plate capacitor formed by the electrodes (of area bw) with the quartz slab (of thickness h and permittivity

ε_T) serving as the dielectric medium. It is therefore given by $\varepsilon_T bw/h$.

(B47)*

Solution- Theoretical Question 3

Part A

Neutrino Mass and Neutron Decay

(a) Let $(c^2E_e, c\vec{q}_e)$, $(c^2E_p, c\vec{q}_p)$, and $(c^2E_\nu, c\vec{q}_\nu)$ be the energy-momentum 4-vectors of the electron, the proton, and the anti-neutrino, respectively, in the rest frame of the neutron. Notice that $E_e, E_p, E_\nu, \vec{q}_e, \vec{q}_p, \vec{q}_\nu$ are all in units of mass.

The proton and the anti-neutrino may be considered as forming a system of total rest mass M_c, total energy c^2E_c, and total momentum $c\vec{q}_c$. Thus, we have

$$E_c = E_p + E_\nu, \qquad \vec{q}_c = \vec{q}_p + \vec{q}_\nu, \qquad M_c^2 = E_c^2 - q_c^2 \qquad \text{(A1)}$$

Note that the magnitude of the vector $\vec{q}_c$ is denoted as q_c. The same convention also applies to all other vectors.

Since energy and momentum are conserved in the neutron decay, we have

$$E_c + E_e = m_n \qquad \text{(A2)}$$

$$\vec{q}_c = -\vec{q}_e \qquad \text{(A3)}$$

When squared, the last equation leads to the following equality

$$q_c^2 = q_e^2 = E_e^2 - m_e^2 \qquad \text{(A4)}$$

From Eq. (A4) and the third equality of Eq. (A1), we obtain

$$E_c^2 - M_c^2 = E_e^2 - m_e^2 \qquad \text{(A5)}$$

With its second and third terms moved to the other side of the equality, Eq. (A5) may be divided by Eq. (A2) to give

$$E_c - E_e = \frac{1}{m_n}(M_c^2 - m_e^2) \qquad \text{(A6)}$$

As a system of coupled linear equations, Eqs. (A2) and (A6) may be solved to give

$$E_c = \frac{1}{2m_n}(m_n^2 - m_e^2 + M_c^2) \qquad \text{(A7)}$$

$$E_e = \frac{1}{2m_n}(m_n^2 + m_e^2 - M_c^2) \qquad \text{(A8)}$$

Using Eq. (A8), the last equality in Eq. (A4) may be rewritten as

$$\begin{aligned} q_e &= \frac{1}{2m_n}\sqrt{(m_n^2 + m_e^2 - M_c^2)^2 - (2m_n m_e)^2} \\ &= \frac{1}{2m_n}\sqrt{(m_n + m_e + M_c)(m_n + m_e - M_c)(m_n - m_e + M_c)(m_n - m_e - M_c)} \end{aligned} \qquad \text{(A9)}$$

Eq. (A8) shows that a maximum of E_e corresponds to a minimum of M_c^2. Now the rest mass M_c is the total energy of the proton and anti-neutrino pair in their center of mass (or momentum) frame so that it achieves the minimum

$$M = m_p + m_\nu \tag{A10}$$

when the proton and the anti-neutrino are both at rest in the center of mass frame. Hence, from Eqs. (A8) and (A10), the maximum energy of the electron $E = c^2E_e$ is

$$E_{\max} = \frac{c^2}{2m_n}\left[m_n^2 + m_e^2 - (m_p + m_\nu)^2\right] \approx 1.292569 \text{ MeV} \approx 1.29\,\text{MeV} \qquad \text{(A11)*}^1$$

When Eq. (A10) holds, the proton and the anti-neutrino move with the same velocity v_m of the center of mass and we have

$$\frac{v_m}{c} = \left(\frac{q_\nu}{E_\nu}\right)\Big|_{E=E_{\max}} = \left(\frac{q_p}{E_p}\right)\Big|_{E=E_{\max}} = \left(\frac{q_c}{E_c}\right)\Big|_{E=E_{\max}} = \left(\frac{q_e}{E_c}\right)\Big|_{M_c=m_p+m_\nu} \tag{A12}$$

where the last equality follows from Eq. (A3). By Eqs. (A7) and (A9), the last expression in Eq. (A12) may be used to obtain the speed of the anti-neutrino when $E = E_{\max}$. Thus, with $M = m_p + m_\nu$, we have

$$\frac{v_m}{c} = \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \qquad \text{(A13)*}$$

$$\approx 0.00126538 \approx 0.00127$$

[Alternative Solution]

Assume that, in the rest frame of the neutron, the electron comes out with momentum $c\vec{q}_e$ and energy c^2E_e, the proton with $c\vec{q}_p$ and c^2E_p, and the anti-neutrino with $c\vec{q}_\nu$ and c^2E_ν. With the magnitude of vector $\vec{q}_\alpha$ denoted by the symbol q_α, we have

$$E_p^2 = m_p^2 + q_p^2, \qquad E_\nu^2 = m_\nu^2 + q_\nu^2, \qquad E_e^2 = m_e^2 + q_e^2 \tag{1A}$$

Conservation of energy and momentum in the neutron decay leads to

$$E_p + E_\nu = m_n - E_e \tag{2A}$$

$$\vec{q}_p + \vec{q}_\nu = -\vec{q}_e \tag{3A}$$

When squared, the last two equations lead to

$$E_p^2 + E_\nu^2 + 2E_pE_\nu = (m_n - E_e)^2 \tag{4A}$$

$$q_p^2 + q_\nu^2 + 2\vec{q}_p \cdot \vec{q}_\nu = q_e^2 = E_e^2 - m_e^2 \tag{5A}$$

Subtracting Eq. (5A) from Eq. (4A) and making use of Eq. (1A) then gives

[1] An equation marked with an asterisk contains answer to the problem.

$$m_p^2 + m_\nu^2 + 2(E_p E_\nu - \vec{q}_p \cdot \vec{q}_\nu) = m_n^2 + m_e^2 - 2m_n E_e \qquad (6A)$$

or, equivalently,

$$2m_n E_e = m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2(E_p E_\nu - \vec{q}_p \cdot \vec{q}_\nu) \qquad (7A)$$

If θ is the angle between $\vec{q}_p$ and $\vec{q}_\nu$, we have $\vec{q}_p \cdot \vec{q}_\nu = q_p q_\nu \cos\theta \le q_p q_\nu$ so that Eq. (7A) leads to the relation

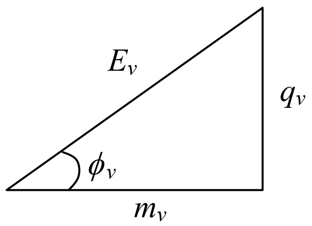
$$2m_n E_e \le m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2(E_p E_\nu - q_p q_\nu) \qquad (8A)$$

Note that the equality in Eq. (8A) holds only if $\theta = 0$, i.e., the energy of the electron $c^2 E_e$ takes on its maximum value only when the anti-neutrino and the proton *move in the same direction*.

Let the speeds of the proton and the anti-neutrino in the rest frame of the neutron be $c\beta_p$ and $c\beta_\nu$, respectively. We then have $q_p = \beta_p E_p$ and $q_\nu = \beta_\nu E_\nu$. As shown in Fig. A1, we introduce the angle ϕ_ν ($0 \le \phi_\nu < \pi/2$) for the antineutrino by

$$q_\nu = m_\nu \tan\phi_\nu, \qquad E_\nu = \sqrt{m_\nu^2 + q_\nu^2} = m_\nu \sec\phi_\nu, \qquad \beta_\nu = q_\nu / E_\nu = \sin\phi_\nu \qquad (9A)$$

Figure A1



Similarly, for the proton, we write, with $0 \le \phi_p < \pi/2$,

$$q_p = m_p \tan\phi_p, \quad E_p = \sqrt{m_p^2 + q_p^2} = m_p \sec\phi_p, \quad \beta_p = q_p / E_p = \sin\phi_p \qquad (10A)$$

Eq. (8A) may then be expressed as

$$2m_n E_e \le m_n^2 + m_e^2 - m_p^2 - m_\nu^2 - 2m_p m_\nu \left(\frac{1 - \sin\phi_p \sin\phi_\nu}{\cos\phi_p \cos\phi_\nu}\right) \qquad (11A)$$

The factor in parentheses at the end of the last equation may be expressed as

$$\frac{1 - \sin\phi_p \sin\phi_\nu}{\cos\phi_p \cos\phi_\nu} = \frac{1 - \sin\phi_p \sin\phi_\nu - \cos\phi_p \cos\phi_\nu}{\cos\phi_p \cos\phi_\nu} + 1 = \frac{1 - \cos(\phi_p - \phi_\nu)}{\cos\phi_p \cos\phi_\nu} + 1 \ge 1 \qquad (12A)$$

and clearly assumes its minimum possible value of 1 when $\phi_p = \phi_\nu$, i.e., when the anti-neutrino and the proton *move with the same velocity* so that $\beta_p = \beta_\nu$. Thus, it follows from Eq. (11A) that the maximum value of E_e is

$$(E_e)_{\max} = \frac{1}{2m_n}(m_n^2 + m_e^2 - m_p^2 - m_v^2 - 2m_p m_v) = \frac{1}{2m_n}[m_n^2 + m_e^2 - (m_p + m_v)^2] \qquad (13A)^*$$

and the maximum energy of the electron $E = c^2 E_e$ is

$$E_{\max} = c^2 (E_e)_{\max} \approx 1.292569 \text{ MeV} \approx 1.29 \text{ MeV} \qquad (14A)^*$$

When the anti-neutrino and the proton move with the same velocity, we have, from Eqs. (9A), (10A), (2A) ,(3A), and (1A), the result

$$\beta_v = \beta_p = \frac{q_p}{E_p} = \frac{q_v}{E_v} = \frac{q_p + q_v}{E_p + E_v} = \frac{q_e}{m_n - E_e} = \frac{\sqrt{E_e^2 - m_e^2}}{m_n - E_e} \qquad (15A)$$

Substituting the result of Eq. (13A) into the last equation, the speed v_m of the anti-neutrino when the electron attains its maximum value $E_{\max}$ is, with $M = m_p + m_v$, given by

$$\frac{v_m}{c} = (\beta_v)_{\max E_e} = \frac{\sqrt{(E_e)_{\max}^2 - m_e^2}}{m_n - (E_e)_{\max}} = \frac{\sqrt{(m_n^2 + m_e^2 - M^2)^2 - 4m_n^2 m_e^2}}{2m_n^2 - (m_n^2 + m_e^2 - M^2)}$$

$$= \frac{\sqrt{(m_n + m_e + M)(m_n + m_e - M)(m_n - m_e + M)(m_n - m_e - M)}}{m_n^2 - m_e^2 + M^2} \qquad (16A)^*$$

$$\approx 0.00126538 \approx 0.00127$$

Part B

Light Levitation

(b) Refer to Fig. B1. Refraction of light at the spherical surface obeys Snell's law and leads to

$$n \sin\theta_i = \sin\theta_t \qquad (B1)$$

Neglecting terms of the order $(\delta/R)^3$ or higher in sine functions, Eq. (B1) becomes

$$n\theta_i \approx \theta_t \qquad (B2)$$

For the triangle ΔFAC in Fig. B1, we have

$$\beta = \theta_t - \theta_i \approx n\theta_i - \theta_i = (n-1)\theta_i \qquad (B3)$$

Let f_0 be the frequency of the incident light. If n_p is the number of photons incident on the plane surface per unit area per unit time, then the total number of photons incident on the plane surface per unit time is $n_p \pi\delta^2$. The total power P of photons incident on the plane surface is $(n_p \pi\delta^2)(hf_0)$, with h being Planck's constant. Hence,

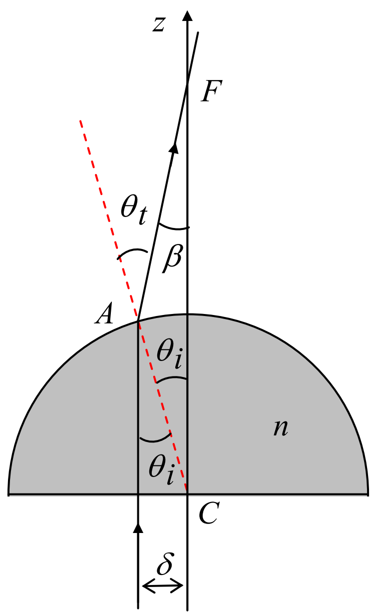


Fig. B1

$$n_p = \frac{P}{\pi\delta^2 hf_0} \qquad \text{(B4)}$$

The number of photons incident on an annular disk of inner radius r and outer radius $r + dr$ on the plane surface per unit time is $n_p(2\pi r dr)$, where $r = R\tan\theta_i \approx R\theta_i$. Therefore,

$$n_p(2\pi r dr) \approx n_p(2\pi R^2)\theta_i d\theta_i \qquad \text{(B5)}$$

The z-component of the momentum carried away per unit time by these photons when refracted at the spherical surface is

$$\begin{aligned} dF_z &= n_p \frac{hf_o}{c}(2\pi r dr)\cos\beta \approx n_p \frac{hf_0}{c}(2\pi R^2)(1-\frac{\beta^2}{2})\theta_i d\theta_i \\ &\approx n_p \frac{hf_0}{c}(2\pi R^2)[\theta_i - \frac{(n-1)^2}{2}\theta_i^3]d\theta_i \end{aligned} \qquad \text{(B6)}$$

so that the z-component of the total momentum carried away per unit time is

$$\begin{aligned} F_z &= 2\pi R^2 n_p (\frac{hf_0}{c})\int_0^{\theta_{im}}[\theta_i - \frac{(n-1)^2}{2}\theta_i^3]d\theta_i \\ &= \pi R^2 n_p (\frac{hf_0}{c})\theta_{im}^2[1-\frac{(n-1)^2}{4}\theta_{im}^2] \end{aligned} \qquad \text{(B7)}$$

where $\tan\theta_{im} = \frac{\delta}{R} \approx \theta_{im}$. Therefore, by the result of Eq. (B5), we have

$$F_z = \frac{\pi R^2 P}{\pi\delta^2 hf_0}(\frac{hf_0}{c})\frac{\delta^2}{R^2}[1-\frac{(n-1)^2\delta^2}{4R^2}] = \frac{P}{c}[1-\frac{(n-1)^2\delta^2}{4R^2}] \qquad \text{(B8)}$$

The force of optical levitation is equal to the sum of the z-components of the forces exerted by the incident and refracted lights on the glass hemisphere and is given by

$$\frac{P}{c} + (-F_z) = \frac{P}{c} - \frac{P}{c}[1-\frac{(n-1)^2\delta^2}{4R^2}] = \frac{(n-1)^2\delta^2}{4R^2}\frac{P}{c} \qquad \text{(B9)}$$

Equating this to the weight mg of the glass hemisphere, we obtain the minimum laser power required to levitate the hemisphere as

$$P = \frac{4mgcR^2}{(n-1)^2\delta^2} \qquad \text{(B10)*}$$